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Linearized gravity in brane backgrounds

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Abstract: A treatment of linearized gravity is given in the Randall-Sundrum background. The graviton propagator is found in terms of the scalar propagator, for which an explicit integral expression is provided. This reduces to the four-dimensional propagator at long distances along the brane, and provides estimates of subleading corrections. Asymptotics of the propagator off the brane yields exponential falloff of gravitational fields due to matter on the brane. This implies that black holes bound to the brane have a "pancake"-like shape in the extra dimension, and indicates validity of a perturbative treatment off the brane. Some connections with the AdS/CFT correspondence are described.

Keywords: Black Holes in String Theory, Classical Theories of Gravity, Beyond Standard Model.

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Contents

1. Introduction and summary

It is possible that the observed world is a brane embedded in a space with more noncompact dimensions. This proposal was made more concrete in the scenario advanced in [1, 2], where the problem of recovering four-dimensional gravity was addressed. (Earlier work appears in refs. [3]–[5].) Further exploration of this scenario has included investigation of its cosmology [6]–[14] and phenomenology [15, 16].[1]

In order to "localize" gravity to the brane, ref. [2] worked in an embedding space with a background cosmological constant, with total action of the form

$$S = \int d^5X\sqrt{-G}(-\Lambda + M^3\mathcal{R}) + \int d^4x\sqrt{-g}\mathcal{L}\,. \tag{1.1}$$

G and X are the five-dimensional metric and coordinates, and g and x are the corresponding four-dimensional quantities with g given as the pullback of the five-dimensional metric to the brane. M is the five-dimensional Planck mass, and $\mathcal{R}$

[1] For a recent survey of some of these topics, see also [17].

denotes the five-dimensional Ricci scalar. The bulk space is a piece of anti-de Sitter space, with radius $R = \sqrt{-12M^3/\Lambda}$, which has metric

$$dS^2 = \frac{R^2}{z^2}(dz^2 + dx_4^2)\,. \tag{1.2}$$

The brane can be taken to reside at $z = R$, or in scenarios [18] with both a probe (or "TeV") brane and a Planck brane, this will be the location of the Planck brane. The horizon for observers on either brane is at $z = \infty$.

There are a number of outstanding questions with this proposal. One very interesting question is what black holes or more general gravitational fields, e.g. due to sources on the brane, look like, both on and off the brane. For example, consider a black hole formed from matter on the brane. From the low-energy perspective of an observer on the brane it should appear like a more-or-less standard four-dimensional black hole but one expects a five-dimensional observer to measure a non-zero transverse thickness. One can trivially find solutions that a four-dimensional observer sees as a black hole by replacing dx^2 with the Schwarzschild metric in (1.2).[2] However, these "tubular" solutions become singular at the horizon at $z = \infty$, suggesting that another solution be found.

Another related question concerns the dynamics of gravity. It was argued in [2] that four-dimensional gravitational dynamics arises from a graviton zero mode bound to the brane. Fluctuations in this zero mode correspond to perturbations of the form

$$dS^2 \to dS^2 + \frac{R^2}{z^2} h_{\mu\nu} dx^\mu dx^\nu\,. \tag{1.3}$$

where $h_{\mu\nu}$ is a function only of x,

$$h_{\mu\nu} = h_{\mu\nu}(x)\,. \tag{1.4}$$

Computing the lagrangian of such a fluctuation yields

$$\mathcal{R} \sim z^2 \partial h \partial h\,. \tag{1.5}$$

This and other measures of the curvature of the fluctuation generically grow without bound as $z \to \infty$. In particular, if we add a higher power of the curvature to the action, with small coefficient, as may be induced from some more fundamental theory of gravity, then generically divergences will be encountered. For example, one easily estimates

$$\mathcal{R}_{\mu\nu\lambda\sigma}\mathcal{R}^{\mu\nu\lambda\sigma} \sim z^4 \tag{1.6}$$

suggesting that Planck scale effects are important near the horizon. This would raise serious questions about the viability of the underlying scenario. These estimates are however incorrect as they neglect the non-zero modes.

[2]Such metrics were independently found in [19].

Yet another question regards corrections to the 4d gravitational effective theory on the brane. We'd like to better understand the strength of corrections to Newton's Law and other gravitational formulae; some of the leading corrections have already been examined via the mode sum [2, 18]. Sufficiently large corrections could provide experimental tests of or constraints on these scenarios.

A final point addresses reinterpretation [20, 21, 22] of these scenarios within the context of AdS/CFT duality [23]–[25]. In this picture, gravity in the bulk AdS off the brane can be replaced by $\mathcal{N} = 4$ super-Yang Mills theory on the brane. Witten [22][3] has suggested that gravitational corrections from the bulk can be reinterpreted in terms of the loop diagrams in the SYM theory.

In order to address these questions, this paper will give an analysis of linearized gravity in the background of [2]. We begin in section 2 with a derivation of the propagator for a scalar field in the brane background of [2], generalized to $d+1$ dimensions. This exhibits much of the physics with less complication than gravity. The propagator is the usual AdS propagator plus a correction term, and can be rewritten, for sources on the brane, in terms of a zero-mode contribution that produces d-dimensional gravity on the brane plus a correction from the "Kaluza-Klein" modes. For even $d \geq 4$ this term produces corrections of order $(R/r)^{d-2}$ at large distances r from the source.

In section 3 we perform a linearization of gravity about the $d+1$ dimensional brane background. For general matter source on the brane, the brane has non-zero extrinsic curvature, and a consistent linear analysis requires introduction of coordinates in which the bending of the brane is manifest, as exhibited in eq. (3.22). We outline the derivation of the graviton propagator, which can be written in terms of the scalar propagator of section 2. (For those readers interested primarily in the applications discussed in the subsequent sections, the results appear in eqs. (3.23), (3.24), and (3.26).) Special simplifying cases include treatment of sources restricted to the Planck brane, or living on a probe brane in the bulk.

In section 4 we discuss the asymptotics and physics of the resulting propagator. Linearized gravity on the Planck brane corresponds to d-dimensional linearized gravity, plus correction terms from Kaluza-Klein modes. As in the scalar case, these yield large-r corrections suppressed by $(R/r)^{d-2}$ for even $d \geq 4$ and by R/r for $d = 3$, in agreement with [27]. We also discuss corrections in the probe-brane scenario of [18]. We then find the falloff in the gravitational potential off the brane and thus deduce the shape in the extra dimensions of black holes bound to the brane or of more general gravitational fields. In particular, we find that black holes have a transverse size that grows with mass like $\log m$, compared to the usual result $m^{1/d-3}$ along the brane. Thus black holes have a pancake-like shape. We also check consistency of the linearized approximation, and check that higher-order curvature terms in the action in fact do not lead to large corrections, as the naïve analysis of the zero mode would indicate.

[3] See also [26].

Finally, section 5 contains some comments on the connection with the AdS/CFT correspondence. An extension of the Maldacena conjecture [23] may enable one to replace the five-dimensional physics off the brane by a suitably regulated large-N gauge theory [20, 21, 22]. In particular, we discuss how this picture produces four-dimensional gravity plus correction terms like those mentioned above.

During the period during which this work has been completed, several related works have appeared. Ref. [27] has found exact solutions for black holes bound to a brane in a $2+1$ dimensional version of this scenario, and in that case independently discovered the shape of black holes (which in fact easily follows from the earlier estimates of [19]). Ref. [28] has also outlined aspects of a linear analysis of gravity, and in particular emphasized the importance of the bending of the brane. Recent comments on the relation with the AdS/CFT correspondence were made by Witten [22], with further elaboration in [26]. Preliminary presentations of some of our results can be found in [29, 30].

2. The massless scalar propagator

Much of the physics of linearized gravity in the scenario of [2] is actually found in the simpler case of a minimally coupled scalar field. Because of this, and because the scalar propagator is needed in order to compute the graviton propagator, this section will focus on computing the scalar Green function.

In much of this paper we will work with the generalization to a $d+1$ dimensional theory with a brane of codimension one. The scalar action, with source terms, takes the form

$$S = \int d^{d+1}X\sqrt{-G}\left[-\frac{1}{2}(\nabla\phi)^2 + J(X)\phi(X)\right]. \tag{2.1}$$

Here we work in the brane background of [2]; for $z > R$ the metric is the $d+1$-dimensional AdS metric,

$$dS^2 = \frac{R^2}{z^2}(dz^2 + dx_d^2)\,. \tag{2.2}$$

Without loss of generality the brane can be located at $z = R$. This problem serves as a toy-model for gravity; for a given source $J(X)$, the resulting field $\phi(X)$ is analogous to the gravitational field of a fixed matter source. The field is given in terms of the scalar Green function, obeying

$$\Box\Delta_{d+1}(X, X') = \frac{\delta^{d+1}(X - X')}{\sqrt{-G}}\,. \tag{2.3}$$

Analogously to the boundary conditions that we will find on the gravitational field, the scalar boundary conditions are taken to be Neumann,

$$\partial_z\Delta_{d+1}(x, x')_{|z=R} = 0\,; \tag{2.4}$$

these can be interpreted as resulting from the orbifold boundary conditions at the brane, or alternately as due to the energy density on the wall. The scalar field has a bound zero mode $\phi = \phi(x)$ analogous to that of gravity.

In order to solve (2.3), we first reduce the problem to solving an ordinary differential equation via a Fourier transform in the d dimensions along the wall,

$$\Delta_{d+1}(x,z;x',z') = \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')} \Delta_p(z,z') \,. \tag{2.5}$$

The Fourier component must then satisfy the equation,

$$\frac{z^2}{R^2}\left(\partial_z{}^2 - \frac{d-1}{z}\partial_z - p^2\right)\Delta_p(z,z') = \left(\frac{z}{R}\right)^{d+1}\delta(z-z') \,. \tag{2.6}$$

Making the definitions $\Delta_p = (zz'/R^2)^{d/2}\hat{\Delta}_p$ and

$$q^2 = -p^2 \,, \tag{2.7}$$

this becomes

$$\left(z^2\partial_z^2 + z\partial_z + q^2z^2 - \frac{d^2}{4}\right)\hat{\Delta}_p(z,z') = Rz\delta(z-z') \,. \tag{2.8}$$

For $z \neq z'$, the equation admits as its two independent solutions the Bessel functions $J_{d/2}(qz)$ and $Y_{d/2}(qz)$. Hence, the solution for $z < z'$ and for $z > z'$ must be linear combinations $\hat{\Delta}_<(z,z')$, $\hat{\Delta}_>(z,z')$ of these functions. Eq. (2.8) then implies matching conditions at $z = z'$:

$$\begin{aligned} \hat{\Delta}_<|_{z=z'} &= \hat{\Delta}_>|_{z=z'} \\ \partial_z(\hat{\Delta}_> - \hat{\Delta}_<)|_{z=z'} &= \frac{R}{z'} \,. \end{aligned} \tag{2.9}$$

We begin with the Green function for $z < z'$. The boundary condition (2.4) translates to

$$\partial_z \left[z^{d/2}\hat{\Delta}_<\right]\Big|_{z=R} = 0 \,. \tag{2.10}$$

This has solution

$$\begin{aligned} \hat{\Delta}_< &= A(z')\left[Y_{d/2-1}(qR)J_{d/2}(qz) - J_{d/2-1}(qR)Y_{d/2}(qz)\right] \\ &= iA(z')\left[J_{d/2-1}(qR)H^{(1)}_{d/2}(qz) - H^{(1)}_{d/2-1}(qR)J_{d/2}(qz)\right] \,, \end{aligned} \tag{2.11}$$

where $H^{(1)} = J + iY$ is the first Hankel function.

Next, consider the region $z > z'$. The boundary conditions at the horizon $z = \infty$ are analogous to the Hartle-Hawking boundary conditions and are inferred by demanding that positive frequency waves be ingoing there, implying [31]

$$\hat{\Delta}_> = B(z')H^{(1)}_{d/2}(qz) \,. \tag{2.12}$$

The matching conditions (2.9) between the regions then become

$$iA(z')\left[J_{d/2-1}(qR)H^{(1)}_{d/2}(qz')-H^{(1)}_{d/2-1}(qR)J_{d/2}(qz')\right]=B(z')H^{(1)}_{d/2}(qz')\,,$$
$$B(z')H^{(1)\prime}_{d/2}(qz')-iA(z')\left[J_{d/2-1}(qR)H^{(1)\prime}_{d/2}(qz')-H^{(1)}_{d/2-1}(qR)J'_{d/2}(qz')\right]=\frac{R}{qz'}\,. \quad (2.13)$$

The solution to these gives

$$\hat{\Delta}_p=i\frac{\pi}{2}R\left[J_{d/2-1}(qR)H^{(1)}_{d/2}(qz_<)-H^{(1)}_{d/2-1}(qR)J_{d/2}(qz_<)\right]\frac{H^{(1)}_{d/2}(qz_>)}{H^{(1)}_{d/2-1}(qR)}\,, \quad (2.14)$$

where $z_>$ ($z_<$) denotes the greater (lesser) of z and z'. This leads to the final expression for the scalar propagator:

$$\Delta_{d+1}(x,z;x',z')=\frac{i\pi}{2R^{d-1}}(zz')^{d/2}\int\frac{d^dp}{(2\pi)^d}e^{ip(x-x')}\times$$
$$\times\left[\frac{J_{d/2-1}(qR)}{H^{(1)}_{d/2-1}(qR)}H^{(1)}_{d/2}(qz)H^{(1)}_{d/2}(qz')-J_{d/2}(qz_<)H^{(1)}_{d/2}(qz_>)\right]. \quad (2.15)$$

We note that the second term is nothing but the ordinary massless scalar propagator in AdS_{d+1}.

A case that will be of particular interest in subsequent sections is that where one of the arguments of Δ_{d+1} is on the Planck brane, at $z=R$. In this case, the propagator is easily shown to reduce to

$$\Delta_{d+1}(x,z;x',R)=\left(\frac{z}{R}\right)^{d/2}\int\frac{d^dp}{(2\pi)^d}e^{ip(x-x')}\frac{1}{q}\frac{H^{(1)}_{d/2}(qz)}{H^{(1)}_{d/2-1}(qR)}\,. \quad (2.16)$$

For both points at $z=R$, a Bessel recursion relation gives a more suggestive result:

$$\Delta_{d+1}(x,R;x',R)=\int\frac{d^dp}{(2\pi)^d}e^{ip(x-x')}\left[\frac{d-2}{q^2R}-\frac{1}{q}\frac{H^{(1)}_{d/2-2}(qR)}{H^{(1)}_{d/2-1}(qR)}\right]. \quad (2.17)$$

This can clearly be separated into the standard d-dimensional scalar propagator Δ_d, with

$$\partial_\mu\partial^\mu\Delta_d(x,x')=\delta^d(x-x')\,, \quad (2.18)$$

which is produced by the zero-mode, plus a piece due to exchange of Kaluza-Klein states:

$$\Delta_{d+1}(x,R;x',R)=\frac{d-2}{R}\Delta_d(x,x')+\Delta_{KK}(x,x')\,. \quad (2.19)$$

Here

$$\Delta_{KK}(x,x')=-\int\frac{d^dp}{(2\pi)^d}e^{ip(x-x')}\frac{1}{q}\frac{H^{(1)}_{d/2-2}(qR)}{H^{(1)}_{d/2-1}(qR)}\,. \quad (2.20)$$

Note that for $d = 3$, this gives the very simple result

$$\Delta_{KK} = i \int \frac{d^3p}{(2\pi)^3} \frac{e^{ip(x-x')}}{q} = \frac{1}{2\pi^2 |x - x'|^2} \,. \tag{2.21}$$

(One must however be careful in treating the gravitational field as a perturbation in this case; recall that in $d = 3$ the potential is logarithmic, which is in a sense not a small perturbation on a Minkowski background.)

The effective action for exchange of ϕ fields between two sources as in (2.1) is given by the usual quadratic expression involving this propagator. While we'll postpone general discussion of the asymptotics of these expressions for large or small x or z until we discuss the graviton propagator, it is worth noting that at large distances on the brane, $|x - x'| \gg R$, the zero mode piece dominates and we reproduce the standard effective action for d-dimensional scalar exchange, plus subleading corrections from the Kaluza-Klein part.

3. Linearized gravity

3.1 General matter source

We next turn to a treatment of linearized gravity in the context of brane reduction. We again work with a $d + 1$-dimensional theory, with action

$$S = \int d^{d+1}X \sqrt{-G} (M^{d-1} \mathcal{R} - \Lambda + \mathcal{L}_{\text{matter}}) - \int d^d x \sqrt{-g}\, \tau \,. \tag{3.1}$$

Here $\mathcal{L}_{\text{matter}}$ may include both matter in the bulk and restricted to the brane. We have explicitly separated out the brane tension τ from the matter lagrangian. Away from the Planck brane, the vacuum solution is $d + 1$-dimensional AdS space, with metric (2.2). The AdS radius is determined by solving Einstein's equations. Denote the Einstein tensor by $\mathcal{G}_{IJ}$; these then take the form

$$\mathcal{G}_{IJ} = \frac{1}{2M^{d-1}} \left[T_{IJ} - \Lambda G_{IJ} - \tau P_{IJ} \delta(X^{d+1} - X^{d+1}(x)) \right] , \tag{3.2}$$

where P_{IJ} is the projection operator parallel to the brane, given in terms of unit normal n^I as

$$P_{IJ} = G_{IJ} - n_I n_J \,, \tag{3.3}$$

and $X^{d+1}(x)$ gives the position of the brane in terms of its intrinsic coordinates x^μ. Off the brane, (3.2) gives

$$R = \sqrt{\frac{-d(d-1)M^{d-1}}{\Lambda}} \,. \tag{3.4}$$

As in [2], the brane tension is fine-tuned to give a Poincare-invariant solution with symmetric (orbifold) boundary conditions about the brane; this condition is

$$\tau = \frac{4(d-1)M^{d-1}}{R} \,. \tag{3.5}$$

The location of the brane is arbitrary; we take it to be $z = R$.

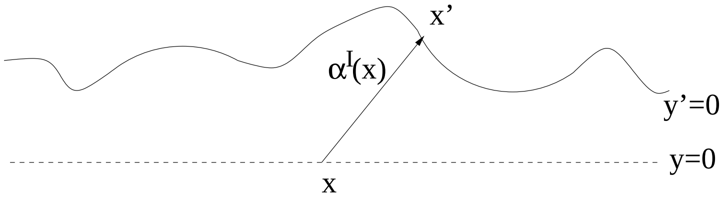

Figure 1: A generic deformation of the base surface leads to a redefinition of gaussian normal coordinates.

The rest of this subsection will focus on deriving the linearized gravitational field due to an arbitrary source; the results are presented in eqs. (3.23), (3.24), and (3.26) for readers not wishing to follow the details of the derivation. As we'll see, maintaining the linearized approximation requires choosing a gauge in which the brane is bent, with displacement given in eq. (3.22).

It is often easier to work with the coordinate y, defined by

$$z = Re^{y/R}\,, \tag{3.6}$$

in which the AdS metric takes the form

$$ds^2 = dy^2 + e^{-2|y|/R}\eta_{\mu\nu}dx^\mu dx^\nu\,; \tag{3.7}$$

the brane is at $y = 0$, and we have written the solution in a form valid for all y.

It is convenient to describe fluctuations about (3.7) in Riemann normal (or hypersurface orthogonal) coordinates, which can be locally defined for an arbitrary spacetime metric which then takes the form

$$ds^2 = dy^2 + g_{\mu\nu}(x,y)dx^\mu dx^\nu\,. \tag{3.8}$$

The coordinate y picks out a preferred family of hypersurfaces, y = const. Such coordinates are not unique; the choice of a base hypersurface on which they are constructed is arbitrary. This base hypersurface may be taken to be the brane, but later another choice will be convenient.

In the case where the coordinates are based on the brane, small fluctuations in the metric can be represented as

$$ds^2 = dy^2 + e^{-2|y|/R}[\eta_{\mu\nu} + h_{\mu\nu}(x,y)]dx^\mu dx^\nu\,. \tag{3.9}$$

Parameterize a deformation of the coordinates corresponding to changing the base hypersurface (see figure 1) by

$$y' = y - \alpha^y(x,y)\,; \qquad x^{\mu\prime} = x^{\mu\prime}(x,y) = x^\mu - \alpha^\mu(x,y) \tag{3.10}$$

and consider a small deformation in the sense that α^y is small. Working at $y > 0$, the condition that the metric takes Gaussian normal form (3.8) in the new coordinates is[4]

$$\begin{aligned} 2\partial_{y'}\alpha^y + g_{\mu\nu}\frac{\partial x^\mu}{\partial y'}\frac{\partial x^\nu}{\partial y'} &= 0 \\ \partial_{\mu'}\alpha^y + g_{\mu\nu}\frac{\partial x^\mu}{\partial y'}\frac{\partial x^\nu}{\partial x^{\mu'}} &= 0\,. \end{aligned} \tag{3.11}$$

In the background metric (3.7), at $y \lesssim R$ this has general solution in terms of arbitrary small functions of x:

$$\begin{aligned} \alpha^y &= \alpha^y(x) \\ \alpha^\mu(x,y) &= -\frac{R}{2}\, e^{2y/R}\partial^\mu\alpha^y(x^\nu) + \beta^\mu(x)\,. \end{aligned} \tag{3.12}$$

The corresponding small gauge transformation in the fluctuation $h_{\mu\nu}$ is

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\alpha_\nu + \partial_\nu\alpha_\mu - \frac{2\alpha^y}{R}\,\eta_{\mu\nu}\,. \tag{3.13}$$

It is straightforward to expand Einstein's equations (3.2) in the fluctuation $h_{\mu\nu}$; at linear order the result is

$$2\mathcal{G}^{(1)y}_y = -{}^{(d)}\mathcal{R} - \frac{(d-1)}{R}\,\partial_y h\epsilon(y) = \frac{T^y_y}{M^{d-1}} \tag{3.14}$$

$$2\mathcal{G}^{(1)y}_\mu = \partial_y\partial^\nu(h_{\mu\nu} - h\eta_{\mu\nu}) = \frac{T^y_\mu}{M^{d-1}} \tag{3.15}$$

$$\mathcal{G}^{(1)\mu}_\nu = {}^{(d)}\mathcal{G}^\mu_\nu + \frac{d\epsilon(y)}{2R}\,\partial_y(h^\mu_\nu - \delta^\mu_\nu h) - \frac{1}{2}\,\partial^2_y(h^\mu_\nu - \delta^\mu_\nu h) = T^\mu_\nu\,\frac{e^{2|y|/R}}{2M^{d-1}} \tag{3.16}$$

where $\mathcal{G}^{(1)}$ denotes the linear part of the Einstein tensor. In these and subsequent equations, indices on "small" quantities $h_{\mu\nu}$, $T_{\mu\nu}$, etc. are raised and lowered with $\eta_{\mu\nu}$. ${}^{(d)}\mathcal{R}$ and ${}^{(d)}\mathcal{G}^\mu_\nu$ represent the curvatures of the induced d-dimensional metric of (3.8) (which include the conformal factor and will subsequently be expanded in h), and we have used the definitions

$$\epsilon(y) = \begin{cases} 1 & \text{if } y > 0; \\ -1 & \text{if } y < 0 \end{cases} \tag{3.17}$$

and $h = h^\mu_\mu$.

Boundary conditions on $h_{\mu\nu}$ at the brane are readily deduced by integrating the equation (3.16) from just below to just above the brane — resulting in the Israel matching conditions [32] — and enforcing symmetry under $y \to -y$. If the energy momentum tensor includes a contribution from matter on the brane,

$$T^{\text{brane}}_{\mu\nu} = S_{\mu\nu}(x)\delta(y)\,, \qquad T^{\text{brane}}_{yy} = T^{\text{brane}}_{y\mu} = 0\,, \tag{3.18}$$

[4]Here we have suppressed a subtlety that arises in trying to write a similar expression valid on both sides of the brane due to the discontinuity in the change of coordinates (3.10) across the brane.

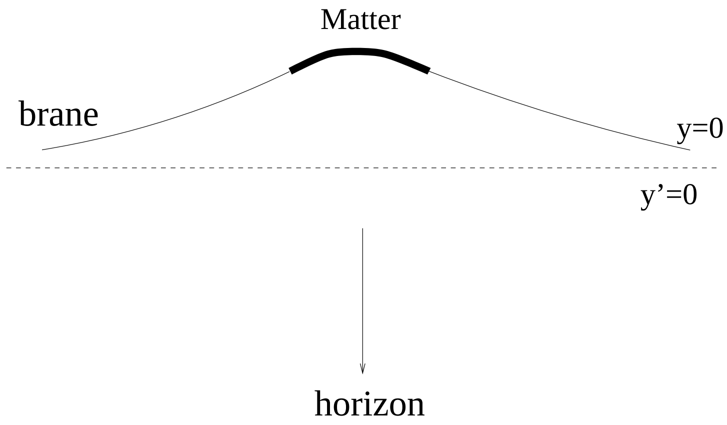

Figure 2: In the presence of matter on the brane, the brane is bent with respect to a coordinate system that is "straight" with respect to the horizon.

then we find

$$\partial_y(h_{\mu\nu} - \eta_{\mu\nu}h)_{|y=0} = -\frac{S_{\mu\nu}(x)}{2M^{d-1}}\,. \tag{3.19}$$

The first step in solving Einstein's equations (3.14)–(3.16) is to eliminate ${}^{(d)}\mathcal{R}$ between the (yy) and $(\mu\mu)$ equations, resulting in an equation for h alone, (working on the $y > 0$ side of the brane)

$$\partial_y\left(e^{-2y/R}\partial_y h\right) = \frac{1}{(d-1)M^{d-1}}\left[T^\mu_\mu - (d-2)e^{-2y/R}\,T^y_y\right]. \tag{3.20}$$

Conservation of T allows this to be rewritten

$$\partial_y\left[e^{-2y/R}\left(\partial_y h + \frac{R}{(d-1)M^{d-1}}T_{yy}\right)\right] = -\frac{R}{(d-1)M^{d-1}}\partial^\mu T_{\mu y}\,. \tag{3.21}$$

This can be integrated with initial condition supplied by the trace of (3.19). There is however an apparent problem if $S^\mu_\mu \neq 0$ or $\partial^\mu T_{\mu y} \neq 0$: the resulting h grows exponentially, leading to failure of the linear approximation.

Fortunately this is a gauge artifact, resulting from basing coordinates on the brane. Indeed, non-vanishing $S_{\mu\nu}(x)$ produces extra extrinsic curvature on the brane; to avoid pathological growth in perturbations one should choose coordinates that are straight with respect to the horizon. In this coordinate system, the brane appears bent, as was pointed out in [28]. (See figure 2). For simplicity consider the case where all matter is localized to $y < y_m$, for some y_m. First, the exponential growth in h due to the initial condition (3.19) can be eliminated by a coordinate transformation of the form (3.12). We may then integrate up in y until we encounter another source for this growth due to $T_{\mu y}$ on the r.h.s. of (3.21), and kill that by again performing a small deformation of the Gaussian-normal slicing. We can proceed iteratively at increasing y in this fashion, with net result that the exponentially growing part of h

can be eliminated by a general slice deformation satisfying

$$\partial_\mu \partial^\mu \alpha^y(x) = \frac{1}{2(d-1)M^{d-1}} \left[\frac{S^\mu_\mu(x)}{2} + RT_{yy}(0) - R\int_0^{y_m} dy \partial^\mu T_{\mu y} \right] ; \tag{3.22}$$

in this equation and the remainder of the section, we work in the region on the $y > 0$ side of the brane. In particular, consider the case $\partial^\mu T_{\mu y} = T_{yy}(0) = 0$; the solution to (3.22) then explicitly gives the bending of the brane due to massive matter on the brane.

In order to solve Einstein's equations we'll therefore work with the metric fluctuation $h'_{\mu\nu}$ in this gauge, which for small coordinate transformations is given by (3.12), (3.13) and (3.22) (the spatial piece $\beta^\mu(x)$ is still arbitrary). Eq. (3.21) has first integral

$$\partial_y h' = -\frac{R}{(d-1)M^{d-1}} \left[T_{yy}(y) - e^{2y/R} \int_y^{y_m} dy \partial^\mu T_{\mu y} \right] \tag{3.23}$$

and can be solved by quadrature, up to the boundary conditions at $y = 0$. Eq. (3.15) is then

$$\partial_y \partial^\nu h'_{\mu\nu} = \partial_y \partial_\mu h' + \frac{T^y_\mu}{M^{d-1}} \tag{3.24}$$

and can be integrated, again given the boundary conditions, to give the longitudinal piece of $h'_{\mu\nu}$. Finally, linearizing ${}^{(d)}\mathcal{G}^\mu_\nu$ in (3.16) and defining

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{3.25}$$

gives

$$\begin{aligned} \Box \bar{h}'_{\mu\nu} = {} & e^{2y/R} \left(-\eta_{\mu\nu} \partial^\lambda \partial^\sigma \bar{h}'_{\lambda\sigma} + \partial^\lambda \partial_\mu \bar{h}'_{\nu\lambda} + \partial^\lambda \partial_\nu \bar{h}'_{\mu\lambda} \right) + \\ & + \frac{\eta_{\mu\nu}}{2} e^{yd/R} \partial_y (e^{-yd/R} \partial_y h') - \frac{e^{2y/R}}{M^{d-1}} T_{\mu\nu} \, . \end{aligned} \tag{3.26}$$

In this expression, $\Box$ is the *scalar* anti-de Sitter laplacian. All quantities on the right-hand side of (3.26) are known, so $h'_{\mu\nu}$ is determined in terms of the scalar Green function for the brane background, found in the preceding section. The metric deformation itself is given by trace-reversing,

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{d-2} \eta_{\mu\nu} \, . \tag{3.27}$$

Note, however, that (3.26) also suffers potential difficulty from exponentially growing sources. By (3.23) we see that for a bounded matter distribution, the trouble only lies in the terms involving $\partial^\nu \bar{h}'_{\mu\nu}$. Note also that outside matter $\partial_y \partial^\nu \bar{h}'_{\mu\nu} = 0$, from (3.24) and (3.23). Therefore, the remaining gauge invariance $\beta^\mu(x)$ in (3.12) can be used to set these contributions to zero just outside the matter distribution,

$$\partial^\nu \bar{h}'_{\mu\nu}|_{y=y_m} = 0 \, , \tag{3.28}$$

and the same holds for all $y > y_m$, eliminating the difficulty. Eq. (3.26) can then be solved for $h'_{\mu\nu}$ using the scalar Green function $\Delta_{d+1}(X, X')$, given in eq. (2.15).

This will give an explicit (but somewhat complicated) formula for the gravitational Green function, defined in general by

$$h_{IJ}(X) = \frac{1}{M^{d-1}} \int d^{d+1}X \sqrt{-G} \Delta_{IJ}^{KL}(X, X') T_{KL}(X') \,, \tag{3.29}$$

and which can be read off in this gauge from (3.23), (3.24) and (3.26). In order to better understand these results, the following two subsections will treat two special cases.

3.2 Matter source on the brane

Consider the case where the only energy-momentum is on the brane. In terms of the flat-space Green function Δ_d, (3.22) determines a brane-bending function of the form

$$\alpha^y(x) = \frac{1}{4(d-1)M^{d-1}} \int d^dx' \, \Delta_d(x, x') \, S^\mu_\mu(x') \,. \tag{3.30}$$

Eq. (3.23) and (3.24) then imply

$$\partial_y h' = 0 = \partial_y \partial^\nu h'_{\mu\nu} \,. \tag{3.31}$$

The gauge freedom $\beta^\mu(x)$ can then be used to set

$$\partial^\nu \bar{h}'_{\mu\nu} = 0 \tag{3.32}$$

and the remaining equation (3.26) becomes

$$\Box \bar{h}'_{\mu\nu} = 0 \,. \tag{3.33}$$

Boundary conditions for this are determined from the boundary condition (3.19) and the gauge shift induced by (3.30), and take the form

$$\partial_y (h'_{\mu\nu} - h'\eta_{\mu\nu})_{|y=0} = -\frac{S_{\mu\nu}}{2M^{d-1}} - 2(\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial_\lambda^2)\, \alpha^y \,. \tag{3.34}$$

In terms of the scalar Neumann Green function Δ_{d+1} of the preceding section, the solution is given by

$$\begin{aligned} h'_{\mu\nu}(X) = \bar{h}'_{\mu\nu}(X) = & -\frac{1}{2M^{d-1}} \int d^dx' \sqrt{-g} \, \Delta_{d+1}(X; x', 0) \times \\ & \times \left[S_{\mu\nu}(x') - \frac{1}{d-1} \eta_{\mu\nu} S(x') + \frac{1}{d-1} \frac{\partial_\mu \partial_\nu}{\partial_\lambda^2} S(x') \right] , \end{aligned} \tag{3.35}$$

where the first equality follows from tracelessness of $\bar{h}'_{\mu\nu}$; the source on the r.h.s. is clearly transverse and traceless as well. Recall that in this gauge the brane is located at $y = -\alpha^y$.

JHEP03(2000)023

The quantity $h'_{\mu\nu}$ is appropriate for discussing observations in the bulk, but a simpler gauge exists for observers on the brane. First note that integration by parts and translation invariance of $\Delta_{d+1}(x,0;x',0)$ implies

$$\int d^dx'\sqrt{-g}\,\Delta_{d+1}(x,0;x',0)\left[2\frac{\partial_\mu\partial_\nu}{\partial_\lambda^2}-\eta_{\mu\nu}\frac{\partial_\lambda^2}{\partial_\lambda^2}\right]S(x')=$$
$$=(2\partial_\mu\partial_\nu-\eta_{\mu\nu}\partial_\lambda^2)\int d^dx'\sqrt{-g}\,\Delta_{d+1}(x,0;x',0)\frac{1}{\partial_\lambda^2}S(x')\,; \qquad (3.36)$$

this, together with a gauge transformation using $\beta^\mu(x)$, can be used to eliminate the third term in (3.35). We then return to $\bar{h}_{\mu\nu}$ by inverting the gauge transformation (3.13); from the d-dimensional perspective the only gauge non-trivial piece is the α^y term, which we rewrite using (3.30). Thus, modulo d-dimensional gauge transformations,

$$\bar{h}_{\mu\nu}(x)=-\frac{1}{2M^{d-1}}\int d^dx'\Big\{\Delta_{d+1}(x,0;x',0)S_{\mu\nu}(x')- \qquad (3.37)$$
$$-\,\eta_{\mu\nu}\left[\Delta_{d+1}(x,0;x',0)-\frac{(d-2)}{R}\Delta_d(x,x')\right]\frac{S^\lambda_\lambda(x')}{2(d-1)}\Big\}.$$

Note from (2.17) that the zero-mode piece cancels in the term multiplying S^λ_λ. Writing the result in terms of the d-dimensional propagator and Kaluza-Klein kernel given in (2.20) then yields

$$\bar{h}_{\mu\nu}(x)=-\frac{d-2}{2RM^{d-1}}\int d^dx'\Delta_d(x,x')S_{\mu\nu}(x')-$$
$$-\frac{1}{2M^{d-1}}\int d^dx'\Delta_{KK}(x,x')\left[S_{\mu\nu}(x')-\frac{\eta_{\mu\nu}}{2(d-1)}S^\lambda_\lambda(x')\right]. \qquad (3.38)$$

The first term is exactly what would be expected from standard d-dimensional gravity, with Planck mass given by[5]

$$M_d^{d-2}=\frac{RM^{d-1}}{d-2}\,. \qquad (3.39)$$

The second term contains the corrections due to the Kaluza-Klein modes.

3.3 Matter source in bulk

As a second example of the general solution provided by (3.26), suppose that the matter source is only in the bulk. This in particular includes scenarios with matter distributions on a probe brane embedded at a fixed y in the bulk [18].

By the Bianchi identities, Einstein's equations are only consistent in the presence of a conserved stress tensor. If we wish to consider matter restricted to a brane at

[5]Our conventions are related to the standard ones for the gravitational coupling G (see e.g. [33]) by $M_4^2=1/16\pi G$ for four dimensions.

JHEP03(2000)023

constant y, a stabilization mechanism[6] must be present to support the matter at this constant "elevation". Consider a stress tensor of the form

$$T_{\mu\nu} = S_{\mu\nu}(x)\delta(y - y_0)\,, \tag{3.40}$$

which is conserved on the brane,

$$\partial^\mu S_{\mu\nu} = 0\,. \tag{3.41}$$

For simplicity assume $T_{\mu y} = 0$. Energy conservation in bulk then states

$$\partial_y \left(e^{-dy/R} T_{yy}\right) = -\frac{e^{(2-d)y/R}}{R} S^\mu_\mu(x)\delta(y - y_0)\,. \tag{3.42}$$

A solution to this with $T_{yy} = 0$ for $y > y_0$ is

$$T_{yy} = \frac{e^{[2y_0+d(y-y_0)]/R}}{R}\theta(y_0 - y)S^\mu_\mu(x)\,. \tag{3.43}$$

We can think of this T_{yy} as arising from whatever physics is responsible for the stabilization.

Whether we consider matter confined to the brane in this way, or free to move about in the bulk, the results of this section give the linearized gravitational solution for a general conserved bulk stress tensor. Assuming for simplicity that $T_{\mu\nu} = T_{yy} = 0$ for $y > y_0$, and that $T_{\mu y} \equiv 0$, we can gauge fix such that (see (3.23) and (3.24))

$$\partial^\mu \bar{h}_{\mu\nu} = 0\,, \qquad \partial_y h = 0\,, \tag{3.44}$$

for $y > y_0$. Thus outside of matter, we see from eq. (3.26) that $h_{\mu\nu}$ again satisfies the scalar AdS wave equation. In particular, for matter concentrated on the probe brane at $y = y_0$, eq. (3.26) gives

$$\Box \bar{h}_{\mu\nu} = -\frac{e^{2y_0/R}}{M^{d-1}}\left[S_{\mu\nu} - \frac{1}{2(d-1)}\eta_{\mu\nu}S^\lambda_\lambda\right]\delta(y - y_0) - \frac{1}{M^{d-1}}\tilde{S}_{\mu\nu}\,, \tag{3.45}$$

where $\tilde{S}_{\mu\nu}$ arises from nonvanishing $\partial^\mu \bar{h}_{\mu\nu}$ for $y < y_0$ on the r.h.s. of (3.26), resulting from the stabilization mechanism. This has solution

$$\begin{aligned}\bar{h}_{\mu\nu}(X) = &-\frac{1}{M^{d-1}}\int d^d x' \sqrt{-g}\,\Delta_{d+1}(X; x', y_0)e^{2y_0/R}\left[S_{\mu\nu}(x') - \frac{1}{2(d-1)}\eta_{\mu\nu}S^\lambda_\lambda(x')\right] - \\ &-\frac{1}{M^{d-1}}\int d^{d+1}X' \sqrt{-G}\,\Delta_{d+1}(X; X')\tilde{S}_{\mu\nu}(X')\,.\end{aligned} \tag{3.46}$$

Again the graviton Green function $\Delta^{KL}_{IJ}(X, X')$ is given in terms of the scalar Green function Δ_{d+1} of (2.15).

[6]See e.g. [34, 35, 14].

4. Asymptotics and physics of the graviton propagator

We now turn to exploration of various aspects of the asymptotic behavior of the propagators given in the preceding two sections, both on and off the brane. This will allow us to address questions involving the strength of gravitational corrections, the shape of black holes, etc.

4.1 Source on "Planck" brane

We begin by examining the gravitational field seen on the "Planck" brane by an observer on the same brane. The relevant linearized field was given in (3.38). This clearly exhibits the expected result from linearized d-dimensional gravity, plus a correction term. The latter gives a subleading correction to gravity at long distances. This can be easily estimated: $x \gg R$ corresponds to $qR \ll 1$, where (2.20) and the small argument formula for the Hankel functions yields

$$\Delta_{KK}(x,x') \approx R\int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}\ln(qR) \propto \frac{R}{r^4} \tag{4.1}$$

for $d=4$, with $r=|x-x'|$. For $d>4$, we need subleading terms in the expansion of the Hankel function. For even d, this takes the general form (neglecting numerical coefficients)

$$H_\nu^{(1)}(x) \sim x^{-\nu}(1+x^2+x^4+\cdots) + x^\nu \ln x(1+x^2+x^4+\cdots)\,. \tag{4.2}$$

Powers of q in the integrand of (2.20) yield terms smaller than powers of $1/r$ (contact terms, exponentially supressed terms). The leading contribution to the propagator comes from the logarithm, with coefficient the smallest power of q. This gives

$$\Delta_{KK}(x,x') \propto \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')} q^{d-4}\ln(qR) \propto \frac{1}{|x-x'|^{2d-4}} \tag{4.3}$$

for $d>4$ and even. For odd d, the logarithm terms are not present in the expansion (4.2), and such corrections vanish. Thus for general even d, the dominant correction terms are supressed by a factor of $(R/r)^{d-2}$ relative to the leading term; these are swamped by post-newtonian corrections. Note that in the special case $d=3$, Δ_{KK} was *exactly* given by eq. (2.21), yielding a correction term of order R/r for a static source, as noted in [27].

One can also examine the short-distance, $r \ll R$, behavior of the propagator, which is governed by the large-q behavior of the Fourier transform. In this case we find

$$\Delta_{KK}(x,x') \approx \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')}\frac{1}{p} \propto \frac{1}{|x-x'|^{d-1}}\,. \tag{4.4}$$

Here clearly the Kaluza-Klein term dominates, and gives the expected $d+1$ dimensional behavior.

Next consider the asymptotics for $z \gg R$ and/or $|x - x'| \gg R$, with a source on the Planck brane. These are dominated by the region of the integral with $q \lesssim \min(1/z, 1/|x-x'|)$. This means that a small argument expansion in qR can be made in the denominator of the propagator (2.16), and this gives

$$\Delta_{d+1}(x, z; x', R) \approx \frac{2\pi i}{R\Gamma\left(\frac{d}{2}-1\right)} \left(\frac{z}{2}\right)^{d/2} \times \\ \times \int \frac{d^d p}{(2\pi)^d} e^{ip(x-x')} q^{d/2-2} H^{(1)}_{d/2}(qz) \left[1 + \mathcal{O}((qR)^{d-2} \log qR)\right]. \quad (4.5)$$

In particular $z \gg |x - x'|$ gives $\exp\{ipx\} \approx 1$, and we find a falloff

$$\Delta_{d+1}(x, z; x', R) \sim \frac{1}{Rz^{d-2}} \quad (4.6)$$

in the propagator at large z.

Note that this means that in the physical case $d = 4$, for a static source, the potential falls like $1/z$ at large z. This calculation can be taken further to determine the asymptotic shape of the Green function and potential as a function of r and z; we do so only for the $d = 4$ potential though the calculation may be extended to other d. The static potential for a source at $x' = 0$ follows from (4.5) by integrating over time,

$$V(x, z) = \int dt \Delta_{4+1}(x, z; 0, R), \quad (4.7)$$

which then becomes

$$V(x, z) \approx \frac{\pi i z^2}{2R} \int \frac{d^3 p}{(2\pi)^3} e^{ipx} H^{(1)}_2(ipz). \quad (4.8)$$

This integral is straightforward to perform, and yields

$$V(x, z) = -\frac{3}{4\pi}\frac{1}{Rz}\left(1 + \frac{2r^2}{3z^2}\right)\left(1 + \frac{r^2}{z^2}\right)^{-3/2}\left[1 + \mathcal{O}\left(\frac{R^2}{z^2}, \frac{R^2}{r^2}\right)\right], \quad (4.9)$$

giving the large-r and large-z dependence. We will return to discuss implications of the $1/z$ behavior shortly.

4.2 Source in bulk or on probe brane

Similar results hold for gravitational sources in the bulk, for example due to matter on a probe brane as described in (3.46). As seen there, the detailed field depends on the form of the stabilization mechanism, but a general understanding follows from consideration of the scalar propagator entering into (3.46). Begin by considering the general scalar propagator (2.15) for $|x - x'| \gg R, z, z'$. This region is again governed by the small-q expansion. At leading order in q, (2.15) again yields

$$\Delta_{d+1}(x, z; x, z') \approx \frac{d-2}{R} \int \frac{d^d p}{(2\pi)^d} e^{ip(x-x')} \frac{1}{q^2}. \quad (4.10)$$

Note that the result is z independent, as in the analogous situation in Kaluza-Klein theory when we consider sources at large separation compared to the compact radius; the long-distance field is determined by the zero mode.

From this result we find that the gravitational potential energy between two objects at coordinate $z > R$ and with $d+1$ dimensional masses m and m' behaves as

$$V(r) \propto \frac{1}{RM^{d-1}} \left(\frac{R}{z}\right)^2 \frac{mm'}{r^{d-3}}\,. \tag{4.11}$$

(Note that this potential only includes contributions from the two objects and not their stabilizing fields.) This can be rewritten in terms of the d-dimensional "physical" mass using

$$m_d = \frac{Rm}{z}\,. \tag{4.12}$$

For probe-brane scenarios [18], it is also important to understand the size of the corrections to this formula. This follows from (2.15) and the expansions (4.2) and (again neglecting numerical coefficients)

$$J_\nu(x) \sim x^\nu(1 + x^2 + x^4 + \cdots)\,. \tag{4.13}$$

Applying these to the first term in (2.15) yields (for even d)

$$\begin{aligned}\frac{J_{d/2-1}(qR)}{H_{d/2-1}(qR)} H^2_{d/2}(qz) \sim \frac{R^{d-2}}{z^d}\frac{1}{q^2}\left[1 + q^2R^2 + q^2z^2 + (qR)^{d-2}\ln(qR) + q^d z^d \ln(qz)\right] \times \\ \times \left\{1 + \mathcal{O}\left[(qz)^2, (qR)^2\right]\right\}.\end{aligned} \tag{4.14}$$

Here we have dropped subdominant terms. The corrections involving simple powers of q again all integrate to yield contact terms at $x = x'$, so the dominant corrections at finite separation come from the logarithmic terms. These then give contributions to the propagator of the form

$$\Delta(x, z; x', z) \sim \frac{1}{Rr^{d-2}}\left[1 + \left(\frac{R}{r}\right)^{d-2} + \frac{z^d}{r^d} + \cdots\right]. \tag{4.15}$$

Combining this with a similar analysis of the second term in (2.15) using

$$J_{d/2}(qz)H_{d/2}(qz) \sim 1 + q^2z^2 + \cdots + q^d z^d \ln(qz) + \cdots \tag{4.16}$$

leads to an expansion of the form

$$\Delta(x, z; x', z') \sim \frac{1}{Rr^{d-2}}\left[1 + \frac{R^{d-2}}{r^{d-2}} + \frac{z^d}{r^d} + \frac{z^{2d}}{r^{2d}}\frac{r^{d-2}}{R^{d-2}}\right]\left[1 + \mathcal{O}\left(\frac{z^2}{r^2}, \frac{R^2}{r^2}\right)\right] \tag{4.17}$$

for the propagator for even d. Which terms are dominant depends on the magnitude of r. At long distances, $r^2 > z^d/R^{d-2}$, the first term is dominant. In the physical case of $d = 4$, the corresponding scale is

$$r \sim \frac{z^2}{R} \sim (10^{-4}\,\mathrm{eV})^{-1} \tag{4.18}$$

in the scenario of [18], and at larger scales the corrections to the newtonian potential therefore go like $1/r^3$ with a Planck-size coefficient, and would be swamped by post-newtonian corrections, as with corrections on the Planck brane. On the other hand, at shorter scales than (4.18), the last term in (4.17) dominates. This gives a propagator correction of the form z^8/r^8 in four dimensions.[7] This is the first correction that we could hope to measure in high-energy experiments.

It is interesting to investigate the asymptotics of the propagator in more detail. Concretely, consider a source at (x', z') with $z' \gg R$. In the limit r^2 and/or $z^2 \gg (z')^d/R^{d-2}$, or $z^d \ll (z')^2 R^{d-2}$, the first term in the propagator (2.15) dominates over the bulk AdS part given by the second term. Consequently, the behavior is given by expressions like (4.5) and (4.9) (where z is replaced by z' for the latter limit). On the other hand, for $r^2, z^2 \ll (z')^d/R^{d-2}$, the bulk AdS portion dominates and hence determines the shape of the potential. This AdS propagator is explicitly given in terms of a hypergeometric function [36, 37]. This transition to bulk AdS behavior is that seen in (4.17). Indeed, the asymptotic behavior of the bulk AdS propagator for $r \gg z, z', R$,

$$G_{\mathrm{AdS}} \sim \left[\frac{zz'}{r^2 + (z-z')^2}\right]^d , \tag{4.19}$$

is what determines the z^{2d}/r^{2d} corrections discussed there.

Finally, the short-distance bulk asymptotics are also easily examined. Specifically, let $|x - x'| \ll R$ and $|z - z'| \ll R$; again, the result is governed by the large-qR behavior of the Green function. From (2.15) we find

$$\Delta_{d+1}(x, z; x', z') \approx -\frac{i\pi}{4R^{d-1}}(zz')^{d/2} \int \frac{d^d p}{(2\pi)^d} e^{ip(x-x')} H^{(1)}_{d/2}(qz_>) H^{(2)}_{d/2}(qz_<) , \tag{4.20}$$

where $H^{(2)} = J - iY$. Aside from non-standard boundary conditions, this is the usual propagator for anti-de Sitter space, and at short distances compared to R will reduce to the flat space propagator (see e.g. [38]). In particular, suppose that $|x - x'| \ll z, z'$. Then we may also use the large-qz expansion, which gives

$$\Delta_{d+1}(x, z; x', z') \approx -\frac{i}{2}\left(\frac{zz'}{R^2}\right)^{\frac{d-1}{2}} \int \frac{d^d p}{(2\pi)^d} e^{ip(x-x') + iq(z_> - z_<)} \frac{1}{q} , \tag{4.21}$$

resulting in

$$\Delta_{d+1}(x, z; x', z') \sim \left(\frac{zz'/R^2}{|x-x'|^2 + |z-z'|^2}\right)^{\frac{d-1}{2}} \tag{4.22}$$

which is the expected behavior in $d + 1$ dimensions.

[7]This is in agreement with [18], and the subleading corrections can also be obtained from the mode sum.

4.3 Off-brane profile of gravitational fields and black holes

These results can readily be applied to discuss some interesting properties of black holes and more general gravitational fields in the context of brane-localized gravity. In particular, one might ask what a black hole — or more general gravitational field — formed from collapsing matter on the brane looks like. In a naïve analysis, considering only fluctuations in the zero mode (1.4), one finds that metrics of the form

$$ds^2 = \frac{R^2}{z^2}[dz^2 + g_{\mu\nu}(x)dx^\mu dx^\nu] \tag{4.23}$$

are solutions of Einstein's equations for general Ricci flat four-dimensional metric $g_{\mu\nu}(x)$. These solutions are, however, singular on the horizon at $z = \infty$, as was discussed in [19]. Nonsingular solutions require excitation of the other modes of the graviton. Although exact solutions are elusive except for $d = 3$ [27], the linear analysis of this paper gives us the general picture.

Consider a massive object, $m \gg 1/R$, on the Planck brane. Without loss of generality its metric may be put in the form (3.9). The linear approximation is valid far from the object (for more discussion, see the next subsection). As we've seen in section 4.1, at long distances along the brane we recover standard linearized gravity, with potential of the form (4.11). Transverse to the brane, we use the expression (4.6), which, with (3.35), implies that

$$h_{00} \sim \frac{m}{M_d^{d-2} z^{d-3}} \tag{4.24}$$

(the extra power of z arises because we are considering the static potential). Note that the proper distance off the brane is given by y, defined in eq. (3.6). Thus in fact the metric due to an object on the brane falls exponentially with proper distance off the brane.

If the mass is compact enough to form a black hole, the corresponding horizon is the surface where (for static black holes) $h_{00}(x, y) = 1$. In the absence of an exact solution, the horizon is approximately characterized by the condition $h_{00} \sim 1$. Since we recover linearized d-dimensional gravity at long distances along the brane, the horizon location on the brane is given by the usual condition in terms of the d-dimensional Planck mass (3.39):

$$r_h^{d-3} \sim \frac{m}{M_d^{d-2}} \,. \tag{4.25}$$

Transverse to the brane (4.24) implies that the horizon is at $z_h \sim r_h$. Taking into account the exponential relation between y and z, a rough measure of the proper "size" of the black hole transverse to the brane is

$$y_h \sim R \log \left[\left(\frac{m}{M_d^{d-2}} \right)^{\frac{1}{d-3}} \frac{1}{R} \right] . \tag{4.26}$$

So while the size along the brane grows like $m^{1/d-3}$, the thickness transverse to the brane grows only like $\log m$. The black hole is shaped like a pancake.[8] This black pancake has a constant coordinate radius r_h in x coordinates for $z \lesssim z_h$, as seen in (4.10). From (2.2) we then see that the proper physical size scales with z as

$$r_h(z) = \frac{R r_h}{z} \,. \tag{4.27}$$

An amusing consequence of this picture of black holes is the possibility that matter on a collision course with a black hole from the four-dimensional perspective can in fact pass around it through the fifth dimension.[9] This suggests that from the perspective of the four-dimensional observer, matter can pass through a black hole! To investigate this more closely, consider the specific case of a black hole on the Planck brane. A bulk mode (graviton or other bulk matter) can bypass the black hole by traveling at $z > z_h$. Suppose first that the proper wavelength of the bulk mode is $\lambda > R$. From the four dimensional perspective of the Planck brane the wavelength is redshifted,

$$\lambda_d \sim \left(\frac{z}{R}\right) \lambda > z_h \left(\frac{\lambda}{R}\right) . \tag{4.28}$$

Thus for $\lambda > R$, such matter has a wavelength larger than the size of the black hole, which suggests that this process is difficult to distinguish from passage around the black hole in four dimensions. Next consider a mode with wavelength $\lambda < R$. Such a perturbation obeys the geometric optics approximation and will fall into the horizon at $z = \infty$; if emitted from the Planck brane it will never return. However, although this particle is moving in the z direction it might be possible that a four-dimensional observer could observe it's projected gravitational field emerging from the far side of the black hole. We leave further investigation of this process for future work.

A semi-quantitative picture of the shape of gravitational fields due to sources in the bulk, e.g. on a probe brane, also follows from the asymptotics of the propagator as described in section 4.2. Again, let the source be at $x' = 0$ and $z' \gg R$. At short distances $r^2, z^2 \ll (z')^d/R^{d-2}$, the shape is given by the bulk AdS propagator (which, as expected, for $r \ll R$ and $z - z' \ll R$ reduces to the flat $d+1$ dimensional result as seen in (4.22).) At longer distances, r^2 and/or $z^2 \gg (z')^d/R^{d-2}$, or $z^d \ll (z')^2 R^{d-2}$, the zero mode begins to dominate with a shape given by (4.9) or its d-dimensional generalization. This determines the pancake shape discussed above.

4.4 Strength of perturbations

Lastly we turn to the question of consistency of both the linear approximation, as well as of the scenario of [2] within the context of a complete theory of quantum

[8] The value of z_h was guessed by [19] who nonetheless referred to the resulting object as a black cigar instead of a black pancake. Ref. [27] also independently found the black pancake picture in the special case of $d = 3$, where they were able to find an exact solution.

[9] We thank L. Susskind for this observation.

gravity, such as string theory (see the next section for comments on the latter). For simplicity we confine the discussion to the $d = 4$ case, although the results clearly extend.

If one were to base one's analysis solely on the properties of the zero-mode of (1.4), serious questions of consistency immediately arise. For one thing, the black tube metrics mentioned in the introduction become singular at the AdS horizon [19]. Alternately, trouble would be encountered when one considered consistency of linearized gravity, or of the overall scenario after higher-order corrections to gravity of the form

$$S \to S + \alpha \int d^5X \sqrt{-G}\mathcal{R}^2 \tag{4.29}$$

are taken into account. The apparent trouble occurs due to the growth of the curvature of the zero-mode correction to the metric, (1.3).

To see the problems more directly, recall that the general structure of the curvature scalar is

$$\mathcal{R} \sim G^{-2}\partial^2 G \sim z^2 \left(\partial^2 h + h\partial^2 h + h^2\partial^2 h + \cdots\right). \tag{4.30}$$

Similarly, scalars comprised of p powers of the curvature will grow like z^{2p}. According to this naïve analysis, such terms would dominate the action of the zero mode, which would suggest the scenario is inconsistent. Likewise, consider the Einstein-Hilbert action,

$$S \sim \int d^5X \frac{1}{z^3}\left(h\partial^2 h + h^2\partial^2 h + \cdots\right). \tag{4.31}$$

Rescaling $h \to z^{3/2}h$ gives an expansion of the form

$$S \sim \int d^5X \left(h\partial^2 h + z^{3/2}h^2\partial^2 h + \cdots\right), \tag{4.32}$$

with diverging expansion parameter, a similar problem. What these arguments do not account for is that far from the brane the zero-mode no longer dominates, and in fact for a static source the perturbation falls as $h \sim 1/z$ as seen in (4.24). This ensures that $\mathcal{R}^p$ corrections and non-linear corrections to linearized Einstein-Hilbert action are indeed suppressed, in accord with the intuition that a localized gravitational source produces a localized field, not a field that is strong at the horizon.

This addresses the question of potential strong gravitational effects near the horizon in the static case. We see that the source of the apparently singular results is the treatment of the zero mode in isolation; with the full propagator the field dies off as it should with z. This leaves open questions about potentially strong effects in dynamical processes. However, once again given the falloff of the propagator with z, we expect this situation to be very similar to that of scattering in the presence of a black hole, where for most local physics processes (e.g. near the brane), the existence of the black hole is irrelevant. However, while we have not attempted to fully describe particles that fall into the AdS horizon, and radiation that is potentially reemitted from the horizon, this may require confronting the usual puzzles of strongly coupled gravity.[10]

5. Relation to the AdS/CFT correspondence

Treatment of the scenario of [2] within string theory in the special case where the bulk space is AdS may naturally incorporate the AdS/CFT correspondence as [20, 21, 22, 26] have recently advocated, although the analogous statement is not known for the case of more general bulk spaces. Here we will make some comments on this and on the connection with our results.

Quantization of the system in the scenario of [2] would be performed by doing functional integrals — or whatever ultimately replaces them in string theory — of the form

$$\int \mathcal{D}\Psi \int \mathcal{D}G e^{iS} \tag{5.1}$$

over configurations near the background (1.2); in this section we will take the brane to reside at a general radius $z = \rho$ and treat it as a boundary. Here the action is of the form (3.1), plus the required surface term $2M^3 \int_\rho d^4x\sqrt{-g}K$ in the presence of the boundary, and Ψ represents all the non-gravitational "matter" fields of string theory, including those describing the matter moving on the brane. In this section we work with $d = 4$, though generalization is straightforward.

Consider the metric part of this integral,

$$Z[\Psi] = \int \mathcal{D}G e^{iS}\,, \tag{5.2}$$

with fixed matter background fields. One may think of doing this functional integral in two steps: first one integrates over all metrics that match a given boundary metric $g_{\mu\nu}$ on the brane, and then one integrates over all such boundary metrics. The latter integral enforces the boundary condition given in (3.19), which in general can be written in terms of the extrinsic curvature $K_{\mu\nu}$ of the brane as

$$(K^\mu_\nu - \delta^\mu_\nu K)_{|z=\rho} = \frac{S^\mu_\nu}{4M^3}\,. \tag{5.3}$$

If one in particular works in the leading semiclassical approximation, this functional integral takes the form

$$Z[\Psi] \approx e^{\frac{i}{4M^3}\int dV dV' T_{IJ}(X)\Delta^{IJ,KL}(X,X')T_{KL}(X')}\,. \tag{5.4}$$

Here T_{IJ} may represent a source either in the bulk or on the brane. $\Delta^{IJ,KL}$ is the gravitational propagator derived in section 3; in particular, as we've seen it obeys the gravitational analog of Neumann boundary conditions.

[10]For related comments see [22].

For sake of illustration, consider the case where the background fields Ψ are only turned on at the brane; also, for simplicity we will ignore fluctuations of matter fields in the bulk. In line with the above comments, we may rewrite

$$Z[\Psi] = \int Dg e^{\frac{i}{2}\int d^4x\sqrt{-g}(\mathcal{L}_{\rm brane}(\Psi)-\tau)} Z[g,\rho] \tag{5.5}$$

where Dg represents the integral over boundary metrics, the bulk functional integral

$$Z[g,\rho] = \int_{G_{|z=\rho}=g} \mathcal{D}G e^{i\int d^5X\sqrt{-G}(M^3\mathcal{R}-\Lambda)+2iM^3\int d^4x\sqrt{-g}K} \tag{5.6}$$

is performed with fixed (Dirichlet) boundary conditions as indicated, and the factor of $1/2$ in (5.5) results from orbifolding to restrict to one copy of AdS instead of two copies on opposite sides of the brane. The latter functional plays a central role in the AdS/CFT correspondence [23], which states [24, 25]

$$\lim_{\rho\to 0} Z[g,\rho] = \left\langle e^{i\int g_{\mu\nu}T^{\mu\nu}} \right\rangle_{\rm CFT}. \tag{5.7}$$

Actually, as it stands, this equation is not quite correct — counterterms must be added to regulate the result as $\rho \to 0$. The infinite part of these counterterms were worked out in [39, 40]; including them, an improved version of the correspondence is

$$\lim_{\rho\to 0} e^{i\int d^4x\sqrt{-g}(b_0+b_2\mathcal{R}+b_4\log(\rho)\mathcal{R}_2)} Z[g,\rho] = \left\langle e^{i\int g_{\mu\nu}T^{\mu\nu}} \right\rangle_{\rm CFT} \tag{5.8}$$

where $b_0 = -6M^3/R$, $b_2 = -RM^3/2$, $b_4 = 2M^3R^3$, and the r.h.s. is given in terms of renormalized quantities, and

$$\mathcal{R}_2 = -\frac{1}{8}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \frac{1}{24}\mathcal{R}^2\,. \tag{5.9}$$

In the present context, a version of the AdS/CFT correspondence for finite ρ is needed. It is natural to conjecture that

$$e^{i\int d^4x\sqrt{-g}(b_0+b_2\mathcal{R}+b_4\log(\rho)\mathcal{R}_2)} Z[g,\rho] = \left\langle e^{i\int g_{\mu\nu}T^{\mu\nu}} \right\rangle_{{\rm CFT},\rho}, \tag{5.10}$$

where ρ defines a corresponding cutoff for the conformal field theory.[11] Related ideas have appeared in a discussion of the holographic renormalization group [41]. We can think of this as providing a definition of the generating functional $Z[g,\rho]$ in terms of that of the cutoff conformal field theory, and thus can rewrite (5.5) as

$$Z[\Psi] = \int Dg e^{i\int d^4x\sqrt{-g}[\frac{1}{2}\mathcal{L}_{\rm brane}(\Psi)-(\tau/2+b_0)-b_2\mathcal{R}-b_4\log(\rho)\mathcal{R}_2]} \left\langle e^{i\int g_{\mu\nu}T^{\mu\nu}} \right\rangle_{{\rm CFT},\rho}. \tag{5.11}$$

[11] One may also expect that finite counterterms could be added by shifting the b_is by terms that vanish as $\rho \to 0$, though the diffeomorphism-invariant origin of these terms is subtle except for b_4. There may also be counterterms involving higher powers of $\mathcal{R}$. We thank V. Balasubramanian and P. Kraus for a discussion on this point.

JHEP03(2000)023

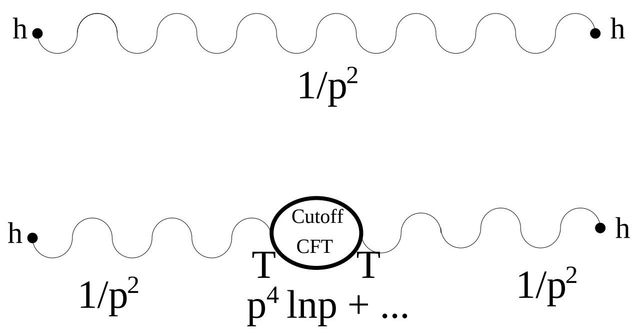

Figure 3: This diagram represents corrections to the graviton propagator due to the cutoff conformal field theory.

The semiclassical/large-N approximation to this may then be compared to (5.4). The parameters τ and b_0 cancel. The curvature term is responsible for the 4-dimensional part of the graviton propagator that was seen in (2.19) and (3.38). The curvature-squared term and the CFT correlators clearly correct this result — from the bulk perspective these corrections arise from the bulk modes which have been integrated out. In particular, the two-point function of the CFT stress tensor,

$$\langle T^{\kappa\lambda} T^{\mu\nu} \rangle_{\mathrm{CFT},\rho} = -\frac{\delta^2}{\delta g_{\kappa\lambda} \delta g_{\mu\nu}} \left\langle e^{i \int g_{\mu\nu} T^{\mu\nu}} \right\rangle_{\mathrm{CFT},\rho} \tag{5.12}$$

gives a leading contribution to the propagator, as argued by Witten [22], and as shown in figure 3. One may readily check that the form of the corrections agrees. From [24], $\langle TT(p) \rangle \sim p^4 \log p$. Attaching two external gravitons gives an extra $1/p^4$, resulting in a correction

$$\Delta_{TT} \sim \int d^4 p e^{ipx} \log p \sim \frac{1}{r^4}\,, \tag{5.13}$$

in agreement with (4.1). A more careful computation could be made to check the coefficient. However, from the above connection, we see that this is not an independent check of the AdS/CFT correspondence: $\langle TT \rangle$ was computed in [24] from supergravity precisely by using the formula (5.7) and regulating, which is simply a rearrangement of the steps in the above discussion. One may in fact relate higher-order terms in the momentum in the full propagator of (2.17) to diagrams with multiple insertions of $\langle TT(p) \rangle_\rho$ and counterterms. An outline of a general argument for this follows from two alternative ways to derive (5.4) from (5.2). On one hand, we could first integrate over the boundary metrics g, finding the constraint (5.3). When we next integrate over the bulk metrics G, we obtain (5.4) with the propagator obeying the analog of Neumann boundary conditions. On the other hand, we could first integrate over the bulk metrics. This will lead to an effective action for g; according to the AdS/CFT prescription (5.11) the kinetic operator in the quadratic term in this action is shifted by $\langle TT(p) \rangle_\rho$. Then integrating over the boundary metric g yields (5.4), where the propagator is the inverse of this shifted kinetic operator.

The suggested connection with the AdS/CFT correspondence provides a potentially interesting new interpretation for the scenario of [2], since the bulk may be completely replaced by the 4d cutoff conformal field theory. There are, however, non-trivial issues in defining the regulated version of this theory, and furthermore for large radius the gravitational description is apparently a more useful approach to computation, since the 't Hooft coupling is large. Notice also that in a sense this is not strictly an induced gravity scenario, since the CFT only induces the corrections due to the bulk modes; eq. (5.11) includes a separate four-dimensional curvature term.

6. Conclusion

The scenario posed in [2] has by now survived several consistency checks, including those of this paper. In a linearized analysis, this paper has outlined many interesting features of gravity and gravitational corrections in this scenario. These await a treatment in an exact non-linear analysis. Many other questions remain, including other aspects of phenomenology and cosmology, and that of a first-principles derivation of such scenarios in string theory.

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